10.

Matrices

- Adding and subtracting matrices
- Multiplying a matrix by a number
- Equal matrices
- Multiplying matrices
- Zero matrices
- Multiplicative identity matrices
- The multiplicative inverse of a square matrix
- Using the inverse matrix to solve systems of equations
- Extension activity: Finding the determinant and inverse of a 3 × 3 matrix
- Miscellaneous exercise ten

Situation

A league soccer competition involves six teams:

Ajax,	Battlers,	Cloggers,
Devils,	Enzymes,	Flames.

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Each team plays one game per week and, during the ten week competition, plays each other team twice, once in the first five weeks and once in the last five weeks. (All teams play at the same venue so no consideration needs to be made to balance home and away games.)

The results for the first five weeks gave rise to the following table:

	Played	Won	Drawn	Lost	Goals scored	
					For	Against
Ajax	5	2	1	2	10	5
Battlers	5	2	1	2	4	5
Cloggers	5	2	0	3	7	6
Devils	5	2	0	3	4	11
Enzymes	5	3	2	0	8	2
Flames	5	2	0	3	5	9

- Create a similar table for the last five weeks of the competition using the results stated below.

Week 6			
Ajax	3	1	Battlers
Cloggers	4	1	Devils
Enzymes	5	4	Flames

Week 7			
Ajax	1	2	Cloggers
Battlers	1	0	Enzymes
Devils	1	1	Flames

Week 8			
Ajax	2	2	Devils
Battlers	1	0	Flames
Cloggers	4	3	Enzymes

Week 9			
Ajax	0	1	Enzymes
Battlers	2	0	Devils
Cloggers	1	0	Flames

Week 10			
Ajax	1	3	Flames
Battlers	0	1	Cloggers
Devils	0	1	Enzymes

- Create a table like the one above for the complete ten week competition.

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As part of the soccer league activity on the previous page we had to arrange information in a 'rows and columns' form of presentation. This rows and columns *rectangular array* presentation of numbers is called a **matrix**. (Plural: matrices).

If we remove the headings and indicate the start and end of the matrix with brackets, the table given on the previous page would be written as shown on the right.

$$\begin{bmatrix} 5 & 2 & 1 & 2 & 10 & 5 \\ 5 & 2 & 1 & 2 & 4 & 5 \\ 5 & 2 & 0 & 3 & 7 & 6 \\ 5 & 2 & 0 & 3 & 4 & 11 \\ 5 & 3 & 2 & 0 & 8 & 2 \\ 5 & 2 & 0 & 3 & 5 & 9 \end{bmatrix}$$

This matrix has 6 rows and 6 columns. We say it is a *six by six* matrix, (written 6×6). This gives the **size** or **dimensions** of the matrix.

Matrices do not have to have the same number of rows as they have columns, however those that do are called **square matrices**.

The matrix on the right is a 6×6 square matrix.

$$\begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & 0.5 \end{bmatrix}$$

2 rows and 3 columns.

A 2×3 matrix.

$$\begin{bmatrix} 2 & 5 \\ 11 & -2 \end{bmatrix}$$

2 rows and 2 columns.

A 2×2 matrix. (A square matrix.)

$$\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

3 rows and 1 column.

A 3×1 matrix.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

3 rows and 4 columns.

A 3×4 matrix.

A matrix consisting of just one column, like the third matrix above, is called a **column matrix**.

Any matrix consisting of just one row is called a **row matrix**.

$$\begin{bmatrix} 5 & 0 & -2 & 1 \end{bmatrix}$$

A square matrix having zeros in all spaces that are not on the **leading diagonal** is called a **diagonal matrix**.

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

We commonly use capital letters to label different matrices. The corresponding lower-case letters, with subscripted numbers, are then used to indicate the row and column a particular entry or **element** occupies.

For the matrix A shown on the right, the element occupying the 3rd row and 2nd column is the number 7.

$$A = \begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 6 & -3 & 9 \\ 5 & 7 & 8 & 4 \end{bmatrix}$$

Thus $a_{32} = 7$.

Similarly $a_{11} = 2$,

$a_{12} = 0$,

$a_{13} = -1$, etc.

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Adding and subtracting matrices

Addition and subtraction of matrices

In the soccer competition activity earlier, you probably determined the matrix for the full ten weeks by adding the matrix for the first five weeks to the matrix for the last five weeks. To perform such addition it was natural to simply add elements occurring in corresponding locations. This is indeed how we add matrices. For example,

$$\text{If } A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & -2 & 3 & 5 \\ 2 & 1 & -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & -3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 2 & 0 & -5 \end{bmatrix} \text{ then } A + B = \begin{bmatrix} 3 & 1 & -1 & 5 \\ 9 & -1 & 5 & 9 \\ 5 & 3 & -3 & -1 \end{bmatrix}$$

$$\text{and similarly} \quad A - B = \begin{bmatrix} -1 & -1 & 5 & 1 \\ -1 & -3 & 1 & 1 \\ -1 & -1 & -3 & 9 \end{bmatrix}$$

Note: When adding or subtracting matrices there must be elements in corresponding spaces. Thus we can only add or subtract matrices that are the same size as each other.

Multiplying a matrix by a number

Suppose that the 3 × 2 matrix shown on the right shows the cost of three models of gas heater in two different shops. Now suppose that in a sale both shops offer 10% discount on all models.

	Shop One	Shop Two
Economy	\$250	\$280
Standard	\$340	\$330
Deluxe	\$450	\$450

The sale prices could be represented in a matrix formed by multiplying each element of the first matrix by 0.9. This is indeed how we multiply a matrix by a number: We multiply each element of the matrix by that number. (This is referred to as 'multiplication by a **scalar**'.)

	Shop One	Shop Two
Economy	\$225	\$252
Standard	\$306	\$297
Deluxe	\$405	\$405

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Equal matrices

For two matrices to be equal, they must be of the same size and have all corresponding elements equal.

Thus if $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 2 & 3 & -5 \\ 1 & 0 & -2 \end{bmatrix}$ then $a = 2 \quad b = 3 \quad c = -5$, $d = 1 \quad e = 0 \quad f = -2$

EXAMPLE 1

If $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ determine each of the following.

If any cannot be determined state this clearly and give the reason.

a A + B **b** A + C **c** B – A **d** 5C **e** 3B – 2A

Solution

a
$$A + B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 7 & 2 \\ 1 & -4 & 3 \end{bmatrix}$$

b A and C are not the same size. Thus A + C cannot be determined.

c
$$B - A = \begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 3 & -6 \\ 1 & 4 & -7 \end{bmatrix}$$

d
$$5C = \begin{bmatrix} 10 & 15 \\ 5 & -25 \end{bmatrix}$$

e
$$3B - 2A = \begin{bmatrix} 9 & 15 & -6 \\ 3 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 8 \\ 0 & -8 & 10 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 11 & -14 \\ 3 & 8 & -16 \end{bmatrix}$$

Many calculators will accept data in matrix form and can then manipulate these matrices in various ways.

Get to know the matrix capability of your calculator.

How does your calculator respond when you ask it to add together two matrices that are not of the same size?

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & -2 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 7 & 2 \\ 1 & -4 & 3 \end{bmatrix}$$

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Exercise 10A

1 A matrix with m rows and n columns has size m × n. Write down the size of each of the following matrices.

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \\ 1 & 0 \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 & -7 & 32 \\ 1 & 1 & 0 & 5 \end{bmatrix} \quad C = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 11 & -2 \end{bmatrix} \quad G = \begin{bmatrix} 12 & 3 \\ 0 & 5 \\ -5 & 2 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2 If e_{mn} is the element situated in the mth row and nth column of matrix E determine

a e_{12} **b** e_{21} **c** f_{13} **d** g_{21} **e** g_{22} **f** g_{32}

where matrices E, F and G are as given below.

$$E = \begin{bmatrix} 5 & 4 & 13 \\ -4 & 2 & 0 \\ 1 & -8 & 12 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 5 & 7 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 2 \\ 7 & 3 \\ -2 & 0 \\ 4 & 11 \end{bmatrix}$$

3 If $A = \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -3 \\ 1 & -5 \end{bmatrix}$ and $D = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$, determine each of the following. (If any cannot be determined state this clearly.)

a A + B **b** A + C **c** C – A **d** 2D
e 3B **f** B + D **g** 2A **h** 2A – C

4 If $P = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 4 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, determine each of the following. (If any cannot be determined state this clearly.)

a P + Q **b** Q – P **c** 3R **d** 3P – 2Q

5 If $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, determine each of the following. (If any cannot be determined state this clearly.)

a A + B **b** 3A **c** B + 2C **d** C + D

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6 If $A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & -3 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 1 & 3 & -1 \\ 2 & 1 & 4 & 3 \\ 1 & 5 & 2 & 0 \end{bmatrix}$, determine each of the following. (If any cannot be determined state this clearly.)

a $A + B$ **b** $A + C$ **c** $2B$ **d** $5A - C$

7 $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 7 \end{bmatrix}$.

For each of the following write 'Yes' if it can be determined and 'No' if it cannot be determined.

a $A + B$ **b** $B - A$ **c** $3C$ **d** $A + D$

e $A - 3D$ **f** $A + 3B$ **g** $B + B$ **h** $A + B + C$

8 Is matrix addition commutative? i.e. Does $A + B = B + A$ (assuming A and B are of the same size)?

9 Is matrix addition associative? i.e. Does $A + (B + C) = (A + B) + C$ (assuming A, B and C are all of the same size)?

10 If $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -7 & 12 \\ 1 & 0 & 13 \end{bmatrix}$, determine matrix C given that the following equation is correct: $3A - 2C = B$.

11 For the first four games in a basketball season the points (P), assists (A), and blocks (B), that five members of one team carried out were as shown below.

Game 1

	P	A	B
Alan	8	5	1
Bob	7	2	4
Dave	14	3	1
Mark	17	3	1
Roger	8	8	2

Game 2

	P	A	B
Alan	12	3	1
Bob	6	8	2
Dave	15	3	5
Mark	6	4	0
Roger	5	2	6

Game 3

	P	A	B
Alan	11	8	2
Bob	15	2	5
Dave	7	5	2
Mark	12	2	1
Roger	14	4	5

Game 4

	P	A	B
Alan	9	4	0
Bob	9	3	3
Dave	11	8	1
Mark	4	12	1
Roger	12	5	3

a Construct a single 5×3 matrix showing the total points, total assists and total blocks each of these five players achieved for the 4 game period.

b Construct a single 5×3 matrix showing the average points per game, average assists per game and average blocks per game for each of these five players for the 4 game period.

12 A company manufactures five types of lawn fertiliser:

- Basic (B)
- Feedit (F)
- Fertilawn (FL)
- Greenit (G)
- Growgrass (GG)

It sells these through its four garden centres (shops).

The number of bags of these fertilisers sold in these centres during the first and second halves of a year are given below:

1 January → 30 June

	B	F	FL	G	GG
Centre I	3100	550	1040	820	2250
Centre II	1640	420	720	480	1480
Centre III	2850	520	1320	640	1250
Centre IV	1240	300	800	360	960

1 July → 31 December

	B	F	FL	G	GG
Centre I	2500	1200	1280	950	2000
Centre II	1200	850	650	540	1240
Centre III	2200	950	1500	640	1450
Centre IV	950	640	720	480	820

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The company predicts that at each shop the sales for the next year will increase by 10% due to a new sales campaign. Assuming this prediction is indeed correct produce a 4×5 matrix showing the number of bags of each fertiliser sold at each shop for the following 1st January → 31st December.

13 If a_{nm} is the element situated in the nth row and mth column of matrix A write down matrix A given that it is a 3×3 matrix with $a_{nm} = 2n + m$.

14 If a_{nm} is the element situated in the nth row and mth column of matrix A write down matrix A given that it is a 3×4 matrix with $a_{nm} = n^m$.

Multiplying matrices

WS Multiplying matrices

An inter-school sports carnival involves five schools competing in seven sports. In each of these sports, medals, certificates and team points are awarded to teams finishing 1st, 2nd or 3rd.

The 5×3 matrix on the right shows the number of first, second and third places gained by each of the five schools.

	1st Place	2nd Place	3rd Place
School A	1	1	1
School B	3	1	0
School C	0	3	3
School D	1	2	0
School E	2	0	3

Suppose that points are awarded using the points system:

1st	3 points
2nd	2 points
3rd	1 point

The total points scored for each school are:

School A	School B	School C	School D	School E
$1 \times 3 +$	$3 \times 3 +$	$0 \times 3 +$	$1 \times 3 +$	$2 \times 3 +$
$1 \times 2 +$	$1 \times 2 +$	$3 \times 2 +$	$2 \times 2 +$	$0 \times 2 +$
1×1	0×1	3×1	0×1	3×1
6	11	9	7	9

Thus school B finished first with 11 points, followed by schools C and E equal second, school D was fourth and school A was fifth.

Note the way that each row of the 5×3 matrix is 'stood up' to align with the points matrix. This is indeed how we carry out matrix multiplication.

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Matrices can be multiplied together if the number of columns in the first matrix equals the number of rows in the second matrix.

If $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & -3 \end{bmatrix}$, the product AB is found as shown below.

Follow each step carefully to make sure you understand where each element in the final answer comes from.

First spin the 1st row of A to align with 1st column of B, multiply and then add:

Thus $$\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} \mathbf{1} & 2 \\ \mathbf{4} & -1 \\ \mathbf{1} & -3 \end{bmatrix} = \begin{bmatrix} (2)(1)+(1)(4)+(3)(1) & \\ & \end{bmatrix}$$

Continue to use the first row of A, this time going *further across* to align with the 2nd column of B. We similarly go further across to place our answer:

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & \mathbf{2} \\ 4 & \mathbf{-1} \\ 1 & \mathbf{-3} \end{bmatrix} = \begin{bmatrix} 9 & (2)(2)+(1)(-1)+(3)(-3) \\ & \end{bmatrix}$$

Having 'exhausted' the 1st row of A we now move *down* to use the 2nd row and similarly move *down* to place our answer:

$$\begin{bmatrix} 2 & 1 & 3 \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{bmatrix}\begin{bmatrix} \mathbf{1} & 2 \\ \mathbf{4} & -1 \\ \mathbf{1} & -3 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ (0)(1)+(-1)(4)+(2)(1) & \end{bmatrix}$$

Continuing the process:

$$\begin{bmatrix} 2 & 1 & 3 \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{bmatrix}\begin{bmatrix} 1 & \mathbf{2} \\ 4 & \mathbf{-1} \\ 1 & \mathbf{-3} \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ -2 & (0)(2)+(-1)(-1)+(2)(-3) \end{bmatrix}$$

Thus $$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ -2 & -5 \end{bmatrix}$$

Confirm this result using your calculator.

Using your calculator to determine the product of matrices can be useful but if the matrices are not too big you should be able to determine the answers mentally. You would not need to show each step of the process and, with practice, you should be able to write the answer directly, as shown at the top of the next page.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & -3 \end{bmatrix}$$
$$\begin{bmatrix} 9 & -6 \\ -2 & -5 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \end{bmatrix}$$

As was mentioned earlier, this method of matrix multiplication means that:

> Two matrices can be multiplied together provided the number of columns in the first matrix equals the number of rows in the second matrix.

Suppose matrix A has dimensions $m \times n$ and matrix B has dimensions $p \times q$.

- The product $A_{mn} B_{pq}$ can only be formed if $n = p$. In this case AB will have dimensions $m \times q$.
- The product $B_{pq} A_{mn}$ can only be formed if $q = m$. In this case BA will have dimensions $p \times n$.

Note: In the product AB we say that B is *premultiplied* by A
or that A is *postmultiplied* by B

EXAMPLE 2

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ determine each of the following. If any cannot be determined state this clearly and explain why.

a AB **b** BA **c** AC **d** CA **e** B^2

Solution

a $AB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ which cannot be determined because the number of columns in A $(2 \times \mathbf{3}) \neq$ the number of rows in B $(\mathbf{2} \times 2)$.

b $BA = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 8 & 6 \\ 2 & 2 & 9 \end{bmatrix}$

c $AC = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 17 \end{bmatrix}$

d $CA = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 0 \end{bmatrix}$ which cannot be determined because the number of columns in C $(3 \times \mathbf{1}) \neq$ the number of rows in A $(\mathbf{2} \times 3)$.

e $B^2 = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 3 & 4 \end{bmatrix}$

Confirm the above answers using your calculator.

EXAMPLE 3

A manufacturer makes three products A, B and C, each requiring a certain number of units of commodities P, Q, R, S and T. Matrix X below shows the number of units of each commodity required to make one of each product.

$$\begin{array}{l} \\ \text{Product A} \\ \text{Product B} \\ \text{Product C} \end{array} \begin{array}{c} \begin{matrix} P & Q & R & S & T \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 0 & 3 \end{bmatrix} \end{array} = X$$

a Each unit of P, Q, R, S and T costs the manufacturer \$200, \$100, \$50, \$400 and \$300 respectively. Write this information as matrix Y which should be either a column matrix or a row matrix, whichever can form a product with X.

b Form the product referred to in **a** and explain what information it displays.

Solution

a As a column matrix, Y would have dimensions 5×1.

As a row matrix, Y would have dimensions 1×5.

Matrix $X_{3\times 5}$ can form a product with $Y_{5\times 1}$: $X_{3\times 5}\,Y_{5\times 1} = Z_{3\times 1}$

$$\text{Thus } Y = \begin{bmatrix} \$200 \\ \$100 \\ \$50 \\ \$400 \\ \$300 \end{bmatrix} \begin{matrix} \leftarrow \text{Cost of 1 unit of P} \\ \leftarrow \text{Cost of 1 unit of Q} \\ \leftarrow \text{Cost of 1 unit of R} \\ \leftarrow \text{Cost of 1 unit of S} \\ \leftarrow \text{Cost of 1 unit of T} \end{matrix}$$

(The order P, Q, R, S, T being consistent with the order in X.)

b

$$XY = \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \\ 50 \\ 400 \\ 300 \end{bmatrix}$$

$$= \begin{bmatrix} 2000 \\ 1400 \\ 1150 \end{bmatrix} \begin{matrix} \leftarrow \text{total commodity cost (\$) for producing 1 unit of product A} \\ \leftarrow \text{total commodity cost (\$) for producing 1 unit of product B} \\ \leftarrow \text{total commodity cost (\$) for producing 1 unit of product C} \end{matrix}$$

Note: Whilst this chapter has considered

- adding and subtracting matrices,
- multiplying a matrix by a scalar
- multiplying matrices

the concept of dividing one matrix by another is undefined for matrices.

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Exercise 10B

Determine each of the following products. If any are not possible state this clearly and explain why.

1 $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$

2 $\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix}$

3 $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix}$

4 $\begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

5 $\begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix}$

6 $\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

7 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

8 $\begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$

10 $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

11 $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

12 $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$

13 $\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$

14 $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ 2 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix}$

15 $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 5 \\ 5 & 1 & -1 \end{bmatrix}$

16 $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 1 \\ -3 & -2 \end{bmatrix}$

17 $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

18 $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix}$

19 If $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, determine the following:

a AB **b** BA **c** A^2 **d** B^2

20 Multiplication of numbers is commutative, i.e. if x and y represent numbers then xy is always equal to yx. Is matrix multiplication commutative for all pairs of matrices for which the necessary products can be formed? Justify your answer.

21 Provided the necessary products can be formed, matrix multiplication is associative, i.e. (AB)C = A(BC). Verify this for

a $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$

b $A = \begin{bmatrix} 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}.$

22 Provided the necessary sums and products can be formed, the distributive law:

$$A(B + C) = AB + AC$$

holds for matrices. Verify this for

a $A = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}.$

b $A = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$

23 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ and k is a number, prove that

$$(kA)B = A(kB) = k(AB)$$

24 A is a 3 × 2 matrix, B is a 3 × 2 matrix, C is a 2 × 3 matrix and D is a 1 × 3 matrix. State the dimensions of each of the following products. For any that cannot be formed state this clearly.

a AB **b** BA **c** BC **d** CB
e AD **f** DA **g** BCA **h** DAC

25 With $A = \begin{bmatrix} 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, state whether each of the following products can be formed or not.

a AB **b** BA **c** AC **d** CA
e BD **f** DB **g** AD **h** DA

26 If it is possible to form the matrix product AA what can we say about matrix A?

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27 BC is just one product that can be formed using two matrices selected from the three below. List all the other products that could be formed in this way. (The selection of the two matrices can involve the same matrix being selected twice.)

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

28 **a** Premultiply $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ by $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$.

b Postmultiply $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ by $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$.

29 The 5 × 3 matrix shown on the right appeared on an earlier page. It shows the number of first, second and third places gained by each of five schools taking part in an inter-school sports carnival involving seven sports.

Determine the rank order for these schools using the points matrix

	1st Place	2nd Place	3rd Place
School A	1	1	1
School B	3	1	0
School C	0	3	3
School D	1	2	0
School E	2	0	3

a

1st	5 points
2nd	3 points
3rd	1 point

b

1st	4 points
2nd	3 points
3rd	2 points

30 A financial adviser sets up share portfolios for three clients. Each portfolio involves shares in 4 companies with the number of shares as shown below.

	Abel Co.	Big Co.	Con Co.	Down Co.
Client 1	1000	5000	400	270
Client 2	500	8000	500	250
Client 3	500	3000	500	500

Initially the value of each share is:

Abel Co.	$5
Big Co.	50 cents
Con Co.	$12
Down Co.	$10

Two years later the value of each share is:

Abel Co.	$4
Big Co.	60 cents
Con Co.	$20
Down Co.	$10

Use matrix multiplication to determine the value of each client's portfolio at each of these times.

31 A fast food outlet offers, amongst other things, Snack Packs and Family Packs. The contents of each of these are as in the contents matrix below:

	Drink (mL)	Number of Burgers
Each Snack Pack	375	1
Each Family Pack	1250	4

An order comes in for 15 Snack Packs and 10 Family Packs. Use matrix multiplication to determine a matrix that shows the total volume of drink and the total number of burgers this order requires.

32 Three hotels each have single rooms, double rooms and suites. The number of each of these in each hotel is as shown in matrix P below:

	Hotel A	Hotel B	Hotel C
Single	15	5	5
Double	25	25	14
Suite	2	1	3

= Matrix P

The three hotels are all owned by the same company and all operate the same pricing structure as shown in the tariff matrix, Q, shown below:

	Single	Double	Suite
Cost per night	\$75	\$125	\$180

= Matrix Q

a Only one of PQ and QP can be formed. Which one?

b Determine the matrix product from part **a** and explain what information it is that this matrix displays.

c Suppose instead that the tariff matrix were written as a column matrix, R. The matrix product PR could be formed but would it give any useful data? Explain your answer.

33 A carpenter runs a business making three different models of cubby house for children. Each cubby house is made using four different sizes of treated pine timber. The number of metres of each size of timber required for each cubby house is shown below.

	Poles 120 mm diameter	Decking 90 mm × 22 mm	Framing 70 mm × 35 mm	Sheeting 120 mm × 12 mm
Cubby A	3	30	20	40
Cubby B	4	35	25	60
Cubby C	6	40	30	70

We will call this matrix P.

a The carpenter receives an order for 3 type As, 1 type B and 2 type Cs.

Write this information as matrix Q which should be either a row matrix or a column matrix, whichever can form a product with P.

b Determine the product referred to in part **a** and explain what this matrix represents.

c The poles cost \$4 per metre, the decking \$2 per metre, the framing \$3 per metre and the sheeting \$1.50 per metre. Write this information as matrix R which should be either a row matrix or a column matrix, dependent on which will form a product with P. What dimensions would this product matrix be and what information would it display?

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34 A manufacturer makes four different models of a particular product. Matrix D below gives the number of units of commodities A, B and C required to make one of each model type.

$$\begin{array}{r} \text{Commodity A} \\ \text{Commodity B} \\ \text{Commodity C} \end{array} \overset{\begin{array}{cccc} \text{Model I} & \text{Model II} & \text{Model III} & \text{Model IV} \end{array}}{\begin{bmatrix} 2 & 3 & 1 & 2 \\ 20 & 30 & 50 & 40 \\ 2 & 1 & 3 & 2 \end{bmatrix}} = D$$

a Each unit of the commodities A, B and C costs the manufacturer \$800, \$50 and \$1000 respectively. Write this information as matrix E, either a column matrix or a row matrix, whichever can form a product with D.

b Form the product referred to in **a** and explain the information it displays.

35 A manufacturer makes three different models of a particular item. Matrix P below gives the number of minutes in the cutting area, the assembling area and the packing area required to make each model.

$$\begin{array}{r} \text{Each model A} \\ \text{Each model B} \\ \text{Each model C} \end{array} \overset{\begin{array}{ccc} \text{Cutting} & \text{Assembling} & \text{Packing} \end{array}}{\begin{bmatrix} 30 & 20 & 10 \\ 20 & 30 & 10 \\ 40 & 40 & 10 \end{bmatrix}} = P$$

The manufacturer receives orders for 50 As, 100 Bs and 80 Cs.

We could write this as a column matrix, Q: $\begin{bmatrix} 50 \\ 100 \\ 80 \end{bmatrix}$

or as a row matrix, R: $\begin{bmatrix} 50 & 100 & 80 \end{bmatrix}$

Both PQ and RP could be formed but only one of these will contain information likely to be useful.

a Which product is this?

b Form the product.

c Explain the information it gives.

Zero matrices

Any matrix which has every one of its entries as zero is called a **zero matrix**.

Thus the 2×2 zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The 2×3 zero matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The letter O is used to indicate a zero matrix. If it is necessary to indicate that it is the zero matrix of a particular dimension, say 2×2 for example, then we write $O_{2 \times 2}$.

There are obvious parallels between zero in the number system and a zero matrix in matrices.

With x representing a number:

$$x + 0 = x, \qquad 0 + x = x, \qquad x \times 0 = 0, \qquad 0 \times x = 0.$$

With A representing a matrix, and providing the necessary sums and products can be formed:

$$\text{A} + \text{O} = \text{A}, \qquad \text{O} + \text{A} = \text{A}, \qquad \text{A} \times \text{O} = \text{O}, \qquad \text{O} \times \text{A} = \text{O}.$$

Because a zero matrix leaves another matrix 'unchanged under addition', i.e. A + O = A and also O + A = A, a zero matrix is sometimes referred to as an *additive identity matrix*.

In this text we will use the letter O rather than the number 0 (zero) for the zero matrix. The two symbols can easily be confused, especially when handwritten. However this should not cause a problem, and you don't need to take time making some distinction between the handwritten characters, because the context usually makes it obvious whether it is the number zero or a zero matrix that is being referred to.

Note: Care needs to be taken when working with matrices. We must guard against using rules and procedures that apply to numbers but that do not necessarily apply to matrices. For example, we have already seen that under matrix multiplication the matrix product AB is usually not the same as BA. Two more points to watch for are given below.

- With x and y representing numbers, a frequently used result in mathematics is that if $xy = 0$ then either $x = 0$ and/or $y = 0$.

  However, for matrices, if AB = O it is not necessarily the case that A and/or B = O.

  For example, consider $\text{A} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ and $\text{B} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$.

  In this case
  $$\text{AB} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

  Thus AB = O but neither A nor B equal O.

- With x and y representing numbers, if $xy = zy$, for $y \neq 0$, then $x = z$.

  However, for matrices, if AB = CB, B ≠ O, matrix A is not necessarily equal to matrix C. i.e. we cannot simply cancel the Bs.

  For example consider $\text{A} = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$, $\text{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\text{C} = \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}$.

  In this case
  $$\text{AB} = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

  and
  $$\text{CB} = \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}.$$

  Thus AB = CB, B ≠ O, but A ≠ C.

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Multiplicative identity matrices

A multiplicative identity matrix leaves all other matrices unchanged under multiplication (provided the multiplication can be performed).

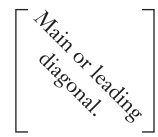
Thus if $I_{m \times n}$ is a multiplicative identity matrix then,

	$I_{m \times n} A_{n \times p} = A_{n \times p}$	from which it follows that m = n.
Also	$B_{q \times m} I_{m \times n} = B_{q \times m}$	from which it follows that m = n.

Thus multiplicative identity matrices are square matrices.

The letter I is used to indicate a multiplicative identity matrix. If clarification is needed as to the size of I we can write I_2 for the 2 × 2 multiplicative identity, I_3 for the 3 × 3 multiplicative identity etc.

A multiplicative identity matrix has every entry of its main or leading diagonal equal to one and every other entry equal to zero.



The 2 × 2 multiplicative identity matrix is: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The 3 × 3 multiplicative identity matrix is: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ etc.

There are obvious parallels between 1 in the number system and I in matrices.

With x representing a number: $1 \times x = x$, $x \times 1 = x$.

With A representing a matrix: $I \times A = A$, $A \times I = A$.

Again be careful not to use rules and procedures that apply to numbers but that do not necessarily apply to matrices.

In numbers, if $xy = x$, then for $x \neq 0$, y must equal 1.

However, for matrices, if $AB = B$, $B \neq O$, A does not necessarily equal I.

For example $\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$.

Premultiplication by $\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$ has left $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ unchanged but $\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} \neq I$.

Thus: Multiplication by I leaves a matrix unchanged, but a matrix being left unchanged under multiplication does not necessarily mean that it must have been I that we multiplied by.

I.e. For matrices A and B, even if we know that AB = B we cannot assume that matrix A = I, the identity matrix.

Thus whilst it is true that:

If matrix A is multiplied by the identity matrix I then A is left unchanged

the converse of this statement is not true.

What about the contrapositive statement?

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WS
Inverse and determinant of a matrix

The multiplicative inverse of a square matrix

The point made on an earlier page:

$$\text{If } AB = CB,\ B \neq O, \text{ then A is not necessarily equal to C}$$

indicates that attempting to divide one matrix by another could present a problem. We might have expected that if $AB = CB$

then 'dividing by B' would give $A = C$.

However, we know that A is not necessarily equal to C.

The direct division of one matrix by another is undefined. However, we can 'undo' the effect of multiplying using the idea of a *multiplicative inverse*.

With numbers, multiplying by $\frac{1}{3}$ $(= 3^{-1})$ undoes the effect of multiplying by 3.

Similarly multiplying by a^{-1} undoes the effect of multiplying by a, $(a \neq 0)$.

We say that a^{-1} is the multiplicative inverse of a.

For $a \neq 0$ we have: $a \times a^{-1} = 1 = a^{-1} \times a$

With matrices, if for some square matrix A there exists a matrix B such that

$$AB = I = BA$$

then we say that B is the multiplicative inverse of A. This is usually simply referred to as the inverse of A, and is written A^{-1}.

Thus $$A\,A^{-1} = I = A^{-1}\,A$$

For example suppose that $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

If $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

from which $\left.\begin{aligned} 3a + 5b &= 1 \\ a + 2b &= 0 \end{aligned}\right\}$ giving $a = 2$ and $b = -1$,

and $\left.\begin{aligned} 3c + 5d &= 0 \\ c + 2d &= 1 \end{aligned}\right\}$ giving $c = -5$ and $d = 3$.

Thus for $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

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Note

- Inverses are only defined for square matrices.
- For the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse A^{-1} is $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$.

  The quantity $(ad - bc)$ is the **determinant** of A, written det A or $|A|$.

  Thus the inverse of a 2×2 matrix can be found as follows:

  1. Interchange the elements of the leading diagonal.
  2. Change the signs of the elements of the other diagonal.
  3. Divide by the determinant.
- The fact that this process involves dividing by the determinant should alert us to the fact that not all 2×2 matrices have inverses. Any matrix with a determinant of zero has no inverse. Such matrices are said to be **singular**.
- A matrix that has an inverse is said to be **invertible**.

EXAMPLE 4

If $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$, determine

a $A + B$ **b** det A **c** A^{-1} **d** B^{-1}

e matrix C such that AC = B **f** matrix D such that DA = B

Solution

a $A + B = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$

b $\det A = (3)(1) - (-1)(2)$
$= 5$

c $A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$ or $\frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

d $B^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{bmatrix}$ or $-\frac{1}{5}\begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}$

e $AC = B$

Premultiply both sides by A^{-1}:

$A^{-1}AC = A^{-1}B$

$IC = A^{-1}B$

$\therefore \quad C = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

f $DA = B$

Postmultiply both sides by A^{-1}:

$DAA^{-1} = BA^{-1}$

$DI = BA^{-1}$

$\therefore \quad D = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix}$

Note that parts **e** and **f** above make use of the fact that for matrices:

$(kP)Q = k(PQ)$ and $P(kQ) = k(PQ)$.

Repeat this example using your calculator.

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EXAMPLE 5

Matrices P, Q and R are all 2 × 2 matrices and $R = P - QR$.

If we assume that $(I + Q)^{-1}$ exists, prove that:

$$R = (I + Q)^{-1} P$$

where I is the 2 × 2 identity matrix.

Hence determine R given that $P = \begin{bmatrix} 20 & 14 \\ 2 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}$.

Solution

Given that $R = P - QR$

then

$$\begin{aligned} R + QR &= P \\ IR + QR &= P \\ (I + Q)R &= P \end{aligned}$$

Premultiply both sides by $(I + Q)^{-1}$

$$(I + Q)^{-1}(I + Q)R = (I + Q)^{-1} P$$

Therefore $R = (I + Q)^{-1} P$ as required.

$$\text{Thus if } P = \begin{bmatrix} 20 & 14 \\ 2 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 20 & 14 \\ 2 & 1 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} 0 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 20 & 14 \\ 2 & 1 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -8 & -4 \\ -16 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

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Exercise 10C

Find the determinants of each of the following 2 × 2 matrices.

1 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **2** $\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ **3** $\begin{bmatrix} -1 & -3 \\ 2 & -1 \end{bmatrix}$ **4** $\begin{bmatrix} -1 & 3 \\ -2 & -1 \end{bmatrix}$

5 $\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ **6** $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ **7** $\begin{bmatrix} x & 0 \\ y & -x \end{bmatrix}$ **8** $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$

Find the inverse of each of the following 2 × 2 matrices. If the matrix does not have an inverse, i.e. the matrix is singular, state this clearly.

9 $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **10** $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$ **11** $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ **12** $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

13 $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ **14** $\begin{bmatrix} -3 & 1 \\ -1 & -3 \end{bmatrix}$ **15** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **16** $\begin{bmatrix} 4 & -3 \\ -8 & 6 \end{bmatrix}$

17 $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ **18** $\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$ **19** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **20** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

21 For each of the following, state whether the given statement is necessarily true for all matrices A, B and C for which the given operations can be determined.

a AI = A **b** IA = A **c** AB = BA
d OA = O **e** $A^{-1}A = I$ **f** $AA^{-1} = I$
g A(B + C) = AB + AC **h** (AB)C = A(BC)
i If AB = O then A = O and/or B = O.
j If AB = AC and A ≠ O then B = C.

For questions **22** to **25** determine matrices A, B, C and D.

22 $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} A = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ **23** $\begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix} B = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$

24 $\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix} C = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ **25** $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

26 Find the constant k given that $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$ and $A^2 - 2A = kI$, where I is the 2 × 2 identity matrix.

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27 Find k if $A = \begin{bmatrix} k & -2 \\ 5 & 0 \end{bmatrix}$ and $A + 10A^{-1} = 5I$, where I is the 2×2 identity matrix.

28 If $A = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 16 & -5 \\ -14 & 5 \end{bmatrix}$ determine

a $A - B$ **b** det B **c** A^{-1} **d** B^{-1}

e matrix C such that $AC = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$ **f** matrix D such that $DA = B$

29 If $P = \begin{bmatrix} -4 & -1 \\ 3 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 6 & 2 \\ 0 & 1 \end{bmatrix}$ determine

a $P + Q$ **b** P^{-1} **c** Q^{-1} **d** $(P + Q)^{-1}$

e matrix R such that $R(P + Q) = Q$

30 Find matrix A given that $AB = \begin{bmatrix} 3 & 1 \\ 22 & 7 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$.

31 Find matrix D given that $CD = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$.

32 For each of the following determine the value(s) of x for which the matrix is singular.

a $\begin{bmatrix} 3x & 4 \\ 6 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} x & 8 \\ 2 & x \end{bmatrix}$ **c** $\begin{bmatrix} x & 2 \\ 10 & (x-1) \end{bmatrix}$

33 Find matrices F and G given that:

$$F = \begin{bmatrix} -2 & 12 \\ 0 & 9 \end{bmatrix}, GE = \begin{bmatrix} -2 & -2 \\ 4 & -2 \end{bmatrix} \text{ and } E^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}.$$

34 Use your calculator to determine matrix C given that:

$$A = \begin{bmatrix} 4 & -2 & -1 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & -6 & -11 \\ 4 & 1 & 1 \\ 6 & 7 & 11 \end{bmatrix} \text{ and } AC = B.$$

35 Use your calculator to determine matrix C given that:

$$A = \begin{bmatrix} 2 & 3 & -3 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & 6 & -7 \\ 1 & 5 & 5 \\ 7 & 11 & -10 \end{bmatrix} \text{ and } CA = B.$$

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36 The entry price matrix, A, for adults and concession card holders (i.e. pensioners and children) attending state and international cricket matches at a particular ground is as follows:

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$$\begin{array}{r} \\ \\ \text{Adult} \\ \text{Concession} \end{array} \begin{array}{cc} \text{State} & \text{International} \\ \text{Match} & \text{Match} \\ \end{array}$$
$$\begin{array}{r} \text{Adult} \\ \text{Concession} \end{array} \begin{bmatrix} \$30 & \$70 \\ \$16 & \$36 \end{bmatrix}$$

To attract more people to matches one proposal under consideration is to reduce all adult non-concessional tickets by 20%, and leave concession tickets unchanged. Under this proposal the new entry price matrix for adults and concession card holders is matrix B, where B has the same row and column meanings as A.

a Write down matrix B.

b Determine matrix C given that CA = B.

37 Matrices A, B and C are all 2×2 matrices and $C = A - CB$. Assuming that $(I + B)^{-1}$ exists, prove that $C = A\,(I + B)^{-1}$ where I is the 2×2 identity matrix.

Hence or otherwise determine C given that $A = \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}$.

38 Matrices A, B and C are all 2×2 matrices and $A = BC - AC$.

Determine C given that $A = \begin{bmatrix} -3 & 5 \\ 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 2 & 4 \end{bmatrix}$.

39 Matrices P and Q are both 2×2 matrices and $P = Q + PQ + PQ^2$.

Determine P given that $Q = \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix}$.

40 A company makes two types of machine, the Oozenplatt and the Flibber. Each Oozenplatt requires six Flantles and five Creks whilst each Flibber requires eight Flantles and seven Creks.

a Copy and complete the following 2×2 matrix, matrix A, for this information.

$$\begin{array}{r} \\ \text{Oozenplatt} \\ \text{Flibber} \end{array} \begin{array}{c} \begin{array}{cc} \text{Flantles} & \text{Creks} \end{array} \\ \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$

The company receives an order for x Oozenplatts and y Flibbers. Matrix B below:

$$\begin{array}{c} \begin{array}{cc} \text{Oozenplatt} & \text{Flibber} \end{array} \\ \begin{bmatrix} x & y \end{bmatrix} \end{array}$$

This order requires a total of 860 Flantles and 740 Creks.

b Write a matrix equation involving A, B and the information about the total numbers of Flantles and Creks required.

c Show how the multiplicative inverse of matrix A can be used to determine x and y (and determine these values in the process).

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Using the inverse matrix to solve systems of equations

Consider the system of linear equations:

$$ax + by = c$$

$$dx + ey = f.$$

This could be written as the matrix equation

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}, \qquad [1]$$

because when we multiply the matrices on the left hand side of this equation we obtain

$$\begin{bmatrix} ax+by \\ dx+ey \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix} \quad \text{i.e.} \quad \begin{matrix} ax+by=c \\ dx+ey=f \end{matrix}$$ the original equations.

The values of x and y can be determined from equation [1] by pre-multiplying both sides by the inverse of $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$.

Note: For the system of equations to have a unique solution the matrix $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ must not be singular. Its determinant must not equal zero.

EXAMPLE 6

a If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, determine A^{-1}.

b Use your answer from **a** to solve $\begin{cases} x - y = 7 \\ 2x + 3y = 4 \end{cases}$.

Solution

a $A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

b First write the equations in matrix form: $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 25 \\ -10 \end{bmatrix}$$

Hence $x = 5$ and $y = -2$.

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EXAMPLE 7

a If $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, use your calculator to determine A^{-1}.

b Use your answer from **a** to solve $\begin{cases} 3x + y + z = -1 \\ 2x - y - z = -4 \\ 2x + y = -5 \end{cases}$.

Solution

a $A^{-1} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.4 & -0.4 & 1 \\ 0.8 & -0.2 & -1 \end{bmatrix}$

or, if you prefer, $\frac{1}{5}\begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 5 \\ 4 & -1 & -5 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & 1 & 0 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.4 & -0.4 & 1 \\ 0.8 & -0.2 & -1 \end{bmatrix}$$

b First express in matrix form: $\begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -5 \end{bmatrix}$.

Thus $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 5 \\ 4 & -1 & -5 \end{bmatrix}\begin{bmatrix} -1 \\ -4 \\ -5 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} -5 \\ -15 \\ 25 \end{bmatrix}$

Hence $x = -1$, $y = -3$ and $z = 5$.

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & 1 & 0 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.4 & -0.4 & 1 \\ 0.8 & -0.2 & -1 \end{bmatrix}$$

$$\text{ans} \times \begin{bmatrix} -1 \\ -4 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}$$

Exercise 10D

Express each of the following in the form $\begin{bmatrix} * & * \\ * & * \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix}$.

1 $\begin{cases} 2x+3y=5 \\ x-3y=0 \end{cases}$

2 $\begin{cases} -x+2y=6 \\ 6x-y=4 \end{cases}$

3 $\begin{cases} 3x+y=-2 \\ x-3y=1 \end{cases}$

Express each of the following in the form $\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$.

4 $\begin{cases} x+y+z=2 \\ 3x-4y+2z=6 \\ x-y-z=4 \end{cases}$

5 $\begin{cases} x+2y+3z=5 \\ 3x-2y=4 \\ 2x-7z=0 \end{cases}$

6 $\begin{cases} 2x-3y+z=1 \\ x+y-3z=0 \\ -2y+3z=4 \end{cases}$

7 a If $A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$, determine A^{-1}.

b Use your answer from **a** to solve $\begin{cases} 3x-2y=4 \\ -5x+4y=-9 \end{cases}$.

8 a If $A = \begin{bmatrix} -2 & 1 & -2 \\ 2 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, use your calculator to determine A^{-1}.

b Use your answer from **a** to solve $\begin{cases} -2x+y-2z=3 \\ 2x-y+3z=-1 \\ y+2z=9 \end{cases}$.

9 Using the idea of a multiplicative inverse of a matrix, and clearly showing that use, solve each of the following systems of equations.

a $\begin{cases} 3x+y=2 \\ 5x+2y=1 \end{cases}$

b $\begin{cases} 3x+y=8 \\ 7x+3y=13 \end{cases}$

10 a If $A = \begin{bmatrix} -2 & -1 & 2 \\ 1 & -1 & 3 \\ 3 & 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 2 & -1 \\ 11 & -2 & 8 \\ 5 & 1 & 3 \end{bmatrix}$, determine AB.

b Express A^{-1} in terms of B.

c Solve the system $\begin{cases} -2x-y+2z=-3 \\ x-y+3z=7 \\ 3x+2y-2z=5 \end{cases}$ clearly showing your use of A^{-1}.

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11 a Express the system of equations shown below left in the form AX = B, as shown below right.

$$\begin{aligned} v + w + x + y + z &= 1 \\ v - w + x + 2y - z &= 13 \\ 2v - w + 3x - y + 2z &= 2 \\ 3v + 2w - x - y - 2z &= 4 \\ 2w + 3y - z &= 8 \end{aligned} \qquad \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \\ * \\ * \end{bmatrix}$$

b Use the ability of your calculator to determine A^{-1}, and to perform matrix multiplication, to solve the system.

Extension activity: Finding the determinant and inverse of a 3 × 3 matrix

Fortunately the ready availability of calculators that can give the determinant and inverse of a square matrix, provided of course that the inverse does exist, means that we no longer have to determine these things 'by hand'.

This extension activity invites you to delve back into history and see how these things were determined prior to the ready availability of today's technology.

Writing the determinant of matrix A as $|A|$, the determinant of the 3×3 matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is defined as

$$|A| = a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Check that this gives the same value for the determinants of

$$\begin{bmatrix} 4 & 1 & 1 \\ 3 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 1 & 2 \\ 5 & 0 & -1 \\ 2 & -3 & 4 \end{bmatrix}$$

as your calculator gives.

To determine the inverse of a 3×3 matrix, determinant $\neq 0$, *in the old days*,

one commonly used method involved 'row reduction'

and another used 'the adjoint matrix'.

Research these methods and write a brief report on each.

Miscellaneous exercise ten

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters in this unit, and the ideas mentioned in the Preliminary work section at the beginning of this unit.

1 If $A = \begin{bmatrix} 2 & 0 \\ -4 & -3 \end{bmatrix}$ find

a matrix B given that A + B = O, the 2 × 2 zero matrix.

b matrix C given that A + C = I, the 2 × 2 identity matrix.

2 If $D = \begin{bmatrix} 5 & -1 \\ 2 & 0 \end{bmatrix}$, O is the 2 × 2 zero matrix and I is the 2 × 2 identity matrix, find

a matrix E given that DO = E,

b matrix F given that D + O = F,

c matrix G given that D + I = G,

d matrix H given that DI = H,

e matrix K given that ID = K.

3 Solve $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$ for $0 \le x \le \pi$.

4 For $0 \le \beta \le \frac{\pi}{2}$ the equation $\tan \beta = k$ has the solution $\beta = p$ radians.

Find all of the solutions to the equation $k \sin 2\theta = 2 \sin^2 \theta$ that exist in the interval $0 \le \theta \le 2\pi$ giving answers in terms of p when it is suitable to do so.

5 Prove that $2 \sin^3 \theta \cos \theta + 2 \cos^3 \theta \sin \theta = \sin 2\theta$.

6 Prove that $\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{1 - \sin 2\theta}{\cos 2\theta}$.

7 **a** Expand $(2y - 1)(y + 1)$.

b Solve: $1 + \sin x = 2 \cos^2 x$ for $-2\pi \le x \le 2\pi$.

8 **a** Express $(2 \cos \theta + 5 \sin \theta)$ in the form $R \cos (\theta - \alpha)$ for α an acute angle in degrees, correct to one decimal place.

b Hence determine the minimum value of $(2 \cos \theta + 5 \sin \theta)$ and the smallest positive value of θ (in degrees correct to one decimal place) for which it occurs.

c Without the assistance of a calculator produce a sketch of the graph of

$$y = 2 \cos x + 5 \sin x$$

for x in degrees.

(Then use a graphic calculator to check the correctness of your sketch.)

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9 If $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, determine each of the following. If any cannot be determined state this clearly and explain why.

a A + B **b** 2A – C **c** AB

d BA **e** AC **f** BD

10 A company makes three models, A, B and C, of a particular item. Each model requires a certain number of units of commodities P, Q and R. Matrix X below shows the number of units of each commodity required to make one of each model.

$$\begin{array}{l} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array} \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \end{array} = X$$

Each unit of P, Q and R costs the company \$50, \$60 and \$200 respectively.

We could write this as a column matrix, Y: $\begin{bmatrix} 50 \\ 60 \\ 200 \end{bmatrix}$ or as a row matrix, Z: $\begin{bmatrix} 50 & 60 & 200 \end{bmatrix}$.

Both XY and ZX could be formed but only one of these will contain information likely to be useful.

a Which is the useful one?

b Form the product.

c Explain the information it displays.

11 Five soccer teams, A, B, C, D and E, take part in a league competition, playing each other once away and once at home. The number of wins, draws and losses of each team are given in matrix X shown on the right.

$$X = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccc} \text{Won} & \text{Drawn} & \text{Lost} \end{array} \\ \begin{bmatrix} 4 & 1 & 3 \\ 3 & 1 & 4 \\ 2 & 3 & 3 \\ 3 & 1 & 4 \\ 5 & 0 & 3 \end{bmatrix} \end{array}$$

Points are awarded:

3 for each win, 1 for each draw and 0 for each loss.

This could be written as a row matrix, matrix Y shown below, or as a column matrix, matrix Z shown below.

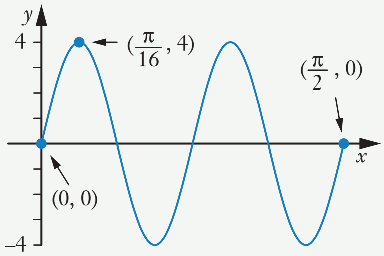
$$Y = \begin{array}{c} \begin{array}{ccc} \text{Win} & \text{Draw} & \text{Loss} \end{array} \\ \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \end{array} \qquad Z = \begin{array}{r} \text{Win} \\ \text{Draw} \\ \text{Loss} \end{array} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

For any of the following products that can be formed, determine the product and explain the information the product matrix displays.

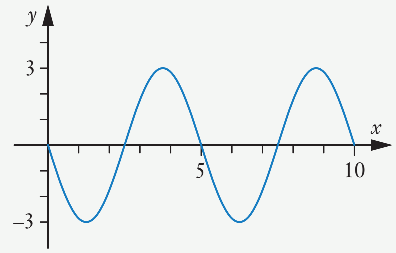
XY, YX, XZ, ZX.

12 Write the equation of each of the following in the form $y = a \sin bx$.

a

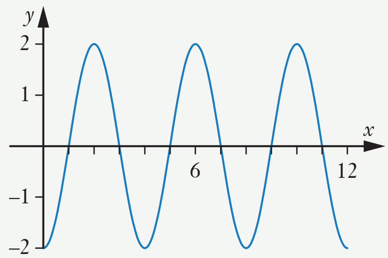


b

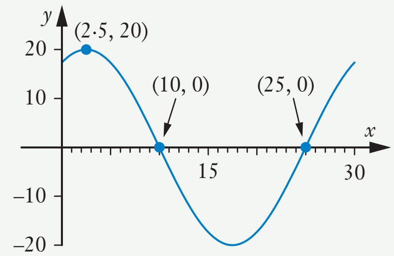


13 Write the equation of each of the following in the form $y = a \sin [b(x + c)]$.

a

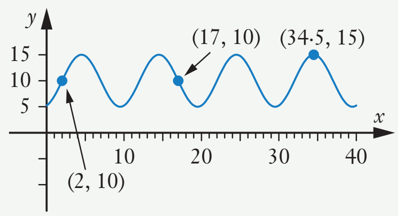


b

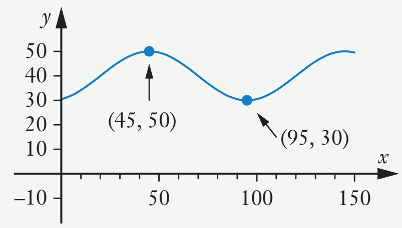


14 Write the equation of each of the following in the form $y = a \sin [b(x + c)] + d$.

a



b



15 Given that $A = \begin{bmatrix} x & 2 \\ y & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ and $AB = BA = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, determine x, y, p, q, r and s.

16 Find matrix P given that $A = \begin{bmatrix} 3 & -1 \\ 8 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}$ and $AP + BP + P = Q$.

17 Matrices A and B are both non singular square matrices.

If A and B commute for multiplication, i.e. AB = BA, prove that

a A and the multiplicative inverse of B commute for multiplication.

b B and the multiplicative inverse of A commute for multiplication.

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